1 Inertial acceleration of the vehicle mass center in body reference frame

Generally, the absolute acceleration of a particle A in a moving reference frame of center B is:

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}$$

Choosing an earth centered reference frame with constant angular velocity ($\vec{\omega}^E$) and no acceleration yield to the simpler form

$$\vec{a}_{A_E} = \vec{\omega}_E^E \times (\vec{\omega}_E^E \times \vec{r}_E^E) + 2\vec{\omega}_E^E \times \vec{v}_E^{rel} + \vec{a}_E^{rel}$$

where the subscript E indicate that the vector are expressed in the earth reference frame.

Transforming this vectorial notation to matrix notation and noting that the angular velocity of F_B relative to F_E is $\vec{\omega}^B - \vec{\omega}^E$ yield to:

$$\vec{a}_{A_E} = \tilde{\omega}_E^E \tilde{\omega}_E^E \vec{r}_E^E + 2\tilde{\omega}_E^E \vec{v}_E^{rel} + \vec{a}_E^{rel}$$

We could now expressed this equation in term of component in the body fised reference frame.

$$\vec{a}_{A_B} = L_{BE}\vec{a}_{A_E} = L_{BE}(\tilde{\omega}_E^E \tilde{\omega}_E^E \vec{r}_E^E + 2\tilde{\omega}_E^E \vec{v}_E^{rel} + \vec{a}_E^{rel})$$

$$= L_{BE}\tilde{\omega}_E^E \tilde{\omega}_E^E \vec{r}_E^E + 2L_{BE}\tilde{\omega}_E^E \vec{v}_E^{rel} + (\tilde{\omega}^B - \tilde{\omega}^E)_B \vec{v}_B^E + \vec{a}_B^{rel}$$

We could then transform

$$L_{BE}\tilde{\omega}_E^E \vec{v}_E^{rel} = (L_{BE}\tilde{\omega}_E^E L_{EB})(L_{BE}\vec{v}_E^{rel}) = \tilde{\omega}_B^E \vec{v}_B^{rel}$$

and

$$L_{BE}\tilde{\omega}_E^E \tilde{\omega}_E^E \vec{r}_E^E = (L_{BE}\tilde{\omega}_E^E L_{EB})(L_{BE}\tilde{\omega}_E^E L_{EB})(L_{BE}\vec{r}_E^E) = \tilde{\omega}_B^E \tilde{\omega}_B^E \vec{r}_B^E$$

We now have

$$\vec{a}_{A_B} = \vec{a}_B^{rel} + (\tilde{\omega}^B - \tilde{\omega}^E)_B \vec{v}_B^E + \tilde{\omega}_B^E \tilde{\omega}_B^E \vec{r}_B^E + \tilde{\omega}_B^E \vec{v}_B^{rel}$$

which simplify to

$$\vec{a}_{A_B} = \vec{a}_B^{rel} + (\tilde{\omega}^B + \tilde{\omega}^E)_B \vec{v}_B^E + \tilde{\omega}_B^E \tilde{\omega}_B^E \vec{r}_B^E$$

Defining vector component in F_B

$$\vec{v}_B^E = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\vec{\omega}_B^B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\vec{\omega}_B^E = \begin{bmatrix} p_B^E \\ q_B^E \\ r_B^E \end{bmatrix} = L_{BV} \begin{pmatrix} \cos\lambda \\ 0 \\ -\sin\lambda \end{pmatrix} \omega^E$$

and matrix notation of angular velocity

$$\tilde{\omega}_B^B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ q & p & 0 \end{bmatrix}$$

$$\tilde{\omega}_B^E = \begin{bmatrix} 0 & -r_B^E & q_B^E \\ r_B^E & 0 & -p_B^E \\ q_B^E & p_B^E & 0 \end{bmatrix}$$

2 Acceleration of vehicle mass center espressed in a spherical coordinate system relative to rotating earth

The spherical coordinate system is compose of the three vector ($\vec{e}_r$, $\vec{e}_\lambda$, $\vec{e}_\mu$), whe $\vec{e}_r$ point in the direction of the mass center of the vehicle, *el* point in the north direction and is perpendicular to $\vec{e}_r$, *eu* point east and is perpendicular to both previous vectors. We could also define the direction of $\vec{e}_r$ with two angle, μ is the angle between the equatorial projection of $\vec{e}_r$ and a reference meridian on the earth. λ is the angle between $\vec{e}_r$ and the equator. We could then define the relative velocity and acceleration of the point C relative to earth by:

$$\vec{v}_{rel} = \dot{r}\vec{e}_r + r\dot{\mu}\cos\lambda\vec{e}_\mu + r\dot{\lambda}\vec{e}_\lambda$$

$$\begin{aligned}\vec{a}_{rel} =&(\ddot{r} - r\dot{\lambda}^2 - r\dot{\mu}^2\cos^2\lambda)\vec{e}_r + \\ &((2\dot{r}\dot{\mu} + r\ddot{\mu})\cos\lambda - 2r\dot{\mu}\dot{\lambda}\sin\lambda)\vec{e}_\mu + \\ &(2\dot{r}\dot{\lambda} + r\ddot{\lambda} + r\dot{\mu}^2\sin\lambda\cos\lambda)\vec{e}_\lambda\end{aligned}$$

We can noe derive the inertial velocity and acceleration of C:

$$\begin{aligned}\vec{v}_C &= \vec{v}_O + \vec{v}_{rel} + \vec{\omega}\times\vec{r} \\ &= \dot{r}\vec{e}_r + r\dot{\mu}\cos\lambda\vec{e}_\mu + r\dot{\lambda}\vec{e}_\lambda + \omega\vec{k}\times r\vec{e}_r \\ &= \dot{r}\vec{e}_r + r\cos\lambda(\dot{\mu}+\omega)\vec{e}_\mu + r\dot{\lambda}\vec{e}_\lambda\end{aligned}$$

$$\vec{a}_C = \vec{a}_O + \vec{\alpha}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\vec{v}_{rel} + \vec{a}_{rel}$$

Each term will be evaluate separatly:

$$\begin{aligned}\vec{\omega}\times(\vec{\omega}\times\vec{r}) &= \omega\vec{k}\times(\omega\vec{k}\times r\vec{e}_r) \\ &= \omega\vec{k}\times(\omega r\cos\lambda\vec{e}_\mu) \\ &= \omega^2 r\cos\lambda(-\cos\lambda\vec{e}_r + \sin\lambda\vec{e}_\lambda \\ &= -\omega^2 r\cos^2\lambda\vec{e}_r + \omega^2 r\cos\lambda\sin\lambda\vec{e}_\lambda\end{aligned}$$

$$\begin{aligned}\vec{\omega}\times\vec{v}_{rel} &= \omega\vec{k}\times(\dot{r}\vec{e}_r + r\dot{\mu}\cos\lambda\vec{e}_\mu + r\dot{\lambda}\vec{e}_\lambda) \\ &= \omega\dot{r}\cos\lambda\vec{e}_\mu + \omega r\dot{\mu}\cos\lambda(-\cos\lambda\vec{e}_r + \sin\lambda\vec{e}_\lambda) + \omega r\dot{\lambda}(-\cos\lambda)\vec{e}_\mu \\ &= \omega\cos\lambda(\dot{r} - r\dot{\lambda})\vec{e}_\mu - \omega r\dot{\mu}\cos^2\lambda\vec{e}_r + \omega r\dot{\mu}\cos\lambda\sin\lambda\vec{e}_\lambda\end{aligned}$$

The inertial acceleration of C is then:

$$\begin{aligned}\vec{a}_C =&(\ddot{r} - r\dot{\lambda}^2 - r\dot{\mu}^2\cos^2\lambda - 2\omega r\dot{\mu}\cos^2\lambda - \omega^2 r\cos^2\lambda)\vec{e}_r \\ &+ ((2\dot{r}\dot{\mu} + r\ddot{\mu})\cos\lambda - 2r\dot{\mu}\dot{\lambda}\sin\lambda + 2\omega\cos\lambda(\dot{r} - r\dot{\lambda})\vec{e}_\mu \\ &+ (2\dot{r}\dot{\lambda} + r\ddot{\lambda} + r\dot{\mu}^2\sin\lambda\cos\lambda + 2\omega r\dot{\mu}\cos\lambda\sin\lambda + \omega^2 r\cos\lambda\sin\lambda)\vec{e}_\lambda\end{aligned}$$

We must now project the inertial acceleration from this vehicled carried frame to the bidy fixed frame with the conversion matrix based on the euler angle which define the angular position of F_B relative to F_V.

$$L_{BV} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

$$
\begin{aligned}
\vec{a}_{C_B} &= L_{BV}\vec{a}_C \\
&= L_{BV}(\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_{rel} + \vec{a}_{rel}) \\
&= L_{BV} \begin{bmatrix} r\cos\lambda\sin\lambda \\ 0 \\ -r\cos^2\lambda \end{bmatrix} \omega^2 + 2\vec{\omega}_B^E \times \vec{v}_{rel_B} + \vec{a}_{rel_B} + (\vec{\omega}^B - \vec{\omega}^E)_B \times \vec{v}_B^E \\
&= L_{BV} \begin{bmatrix} r\cos\lambda\sin\lambda \\ 0 \\ -r\cos^2\lambda \end{bmatrix} \omega^2 + \vec{a}_{rel_B} + (\vec{\omega}^B + \vec{\omega}^E)_B \times \vec{v}_B^E \\
&= \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} + \begin{bmatrix} (q+q_B^E)w - (r+r_B^E)v \\ (r+r_B^E)u - (p+p_B^E)w \\ (p+p_B^E)v - (q+q_B^E)u \end{bmatrix}
\end{aligned}
$$

We will also use the moment equation:

$$
\begin{aligned}
L &= I_x\dot{p} - (I_y - I_z)qr \\
M &= I_y\dot{q} - (I_z - I_x)rp \\
N &= I_z\dot{r} - (I_x - I_y)pq
\end{aligned}
$$

3 System of equation

3.1 Independant variable

(ϕ, θ, ψ)	Euler angle defining orientation of the vehicle relative to F_V
(u, v, w)	Velocity of the vehicle relative to earth
(p, q, r)	Absolute angle rate
(λ, μ, R)	Position of the rocket relative to earth
(p_B^E, q_B^E, r_B^E)	Earth angular velocity in body axis
(P, Q, R)	Relative angular velocity of body relative ro F_V expressed in F_B

3.2 List of equations

$$
\begin{aligned}
X - mg\sin\theta &= m(\dot{u} + c_x + (q_B^E + q)w - (r_B^E + r)v) \\
Y + mg\cos\theta\sin\phi &= m(\dot{v} + c_y + (r_B^E + r)u - (p_B^E + p)w) \\
Z + mg\cos\theta\cos\phi &= m(\dot{w} + c_z + (p_B^E + pvw - (q_B^E + q)u)
\end{aligned}
$$

$$
\begin{aligned}
L &= I_x\dot{p} - (I_y - I_z)qr \\
M &= I_y\dot{q} - (I_z - I_x)rp \\
N &= I_z\dot{r} - (I_x - I_y)pq
\end{aligned}
$$

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}
$$

$$
\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} - L_{BV} \begin{bmatrix} (\omega^E - \dot{\mu})\cos\lambda \\ -\dot{\lambda} \\ -(\omega^E + \dot{\mu})\sin\lambda \end{bmatrix}
$$

$$\begin{bmatrix} \dot{\lambda} r \\ \dot{\mu} r \cos\lambda \\ -\dot{r} \end{bmatrix} = L_{VB} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\vec{\omega}_B^E = \begin{bmatrix} p_B^E \\ q_B^E \\ r_B^E \end{bmatrix} = L_{BV} \begin{pmatrix} \cos\lambda \\ 0 \\ -\sin\lambda \end{pmatrix} \omega^E$$

$$L_{VB} = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$